

ON SIMPLICIAL RESOLUTIONS FOR LINK GROUPS

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This note presents an alternative perspective on the paper [1]. We give two proofs of Theorem 0.1.

Let L be a framed link with ℓ components in S^3 . Then one forms L_i as the i -th cabling (of each component) of L . We write $L = L_0$ and then

L_n consists of $(n+1)\ell$ components.

Put $G_n := \pi_1(S^3 \setminus L_n)$. As explained in [1], these groups form a simplicial group, with d_i induced by the inclusion obtained by forgetting the i -th copy of each component:

$$S^3 \setminus L_{n+1} \hookrightarrow S^3 \setminus L_n.$$

The following theorem is the main result of the paper [1].

Theorem 0.1 (see [1]). *The realization of G_* is weakly equivalent to ΩS^3 .*

For brevity and clarity of the illustrations, we restrict the proofs to the case of a nontrivial knot with trivial framing; that is, $L = K$ with trivial framing.

1. First proof

The first proof follows essentially the same approach as the original authors, using the Quillen spectral sequence. A shortcut makes the argument shorter and arguably more transparent.

Theorem 1.1 (see A.9 in [2]). *For a simplicial group $G \in \mathbf{sGrp}$, there is a natural first-quadrant spectral sequence with*

$$E_{p,q}^1 = H_p(G_q) \implies H_{p+q}(\overline{W}G_*).$$

Moreover, the differential $d_1 : E_{p,q}^1 \rightarrow E_{p,q-1}^1$ is given by $\sum (-1)^i d_i$, using the $\mathbf{sAb}$ structure on $H_p(G_*)$ for each p .

We apply Theorem 1.1 to G_* . The following lemma is proved in [1].

Lemma 1.2. *Each link L_n is non-split. As a consequence of Papakyriakopoulos' theorem, each space $S^3 \setminus L_n$ is a $K(G_n, 1)$.*

Thus $H_p(G_q) = H_p(S^3 \setminus L_q)$. We collect the following standard facts about the topology of $S^3 \setminus L$ in a single lemma.

Lemma 1.3. *Let $L \subset S^3$ be a non-split link with n components. Then:*

- (1) $H_1(S^3 \setminus L) \cong \mathbb{Z}^n$ and is generated by the meridians of the boundary tori corresponding to the components of L .
- (2) $H_2(S^3 \setminus L) \cong \mathbb{Z}^{n-1}$, and is generated by the boundary tori $[T_1], \dots, [T_n]$ corresponding to the components of L , subject to the single relation

$$[T_1] + \dots + [T_n] = 0.$$

- (3) $H_k(S^3 \setminus L) = 0$ for $k \geq 3$.

By Lemma 1.3, only three rows can be nonzero on the E^1 page of the spectral sequence in Theorem 1.1.

Computation of $E_{*,0}^2$ Since all of the spaces are connected, the Δ -group structure of $\{H_0(S^3 \setminus L_n)\}_n$ is that of $\text{const}(\mathbb{Z})$. Thus the first row of E^2 is

$$\mathbb{Z}, 0, 0, 0, \dots$$

Computation of $E_{*,1}^2$ By Lemma 1.3, $H_1(S^3 \setminus L_n) = \mathbb{Z}^{n+1}$, and each face map d_i deletes the i -th generator. The homology of this Δ -group vanishes because the underlying simplicial set is contractible. Thus the second row of E^2 is

$$0, 0, 0, 0, \dots$$

Computation of $E_{*,2}^2$ By Lemma 1.3, $H_2(S^3 \setminus L_n) = \mathbb{Z}^n$; the face maps d_i delete the i -th generators, and d_n deletes the sum of the generators. This Δ -group fits into a short exact sequence with the first two Δ -groups, which at the level of their complexes reads

$$\underbrace{E_{*,0}^1}_A \hookrightarrow \underbrace{E_{*,1}^1}_B \twoheadrightarrow \underbrace{E_{*,2}^1}_C.$$

Thus $H_{\geq 2}$ vanishes. Consider the end of the long exact sequence associated with this short exact sequence:

$$\underbrace{H_1(B)}_0 \rightarrow H_1(C) \rightarrow \underbrace{H_0(A)}_{\mathbb{Z}} \rightarrow \underbrace{H_0(B)}_0 \rightarrow H_0(C).$$

Thus the third row of E^2 is

$$0, \mathbb{Z}, 0, 0, \dots$$

In summary, the second page of the spectral sequence is

	0	$\mathbb{Z}$	0	0
	0	0	0	0
	$\mathbb{Z}$	0	0	0

Thus $H_*(\overline{W}G_*)$ coincides with $H_*(S^3)$. Since $\overline{W}G_*$ is simply connected (Lemma 4.1 in [1]) and 1-connected Moore spaces are unique up to homotopy, G_* is weakly equivalent to ΩS^3 .

2. Second proof

The second proof is geometric. Let N be a small tubular neighbourhood of L . We may assume that each L_n is contained in N . Then $S^3 \setminus L$ can be covered by two open sets:

- (1) $A := N + B_\varepsilon$, where B_ε is a ball of radius ε and the notation denotes a small open thickening of N . Put $A_n = A \cap (S^3 \setminus L_n)$,
- (2) $B := (S^3 \setminus N) + B_\varepsilon$, a small open thickening of $S^3 \setminus N$ using the same ball B_ε . Put $B_n = B \cap (S^3 \setminus L_n)$.

These two sets have a connected intersection, homotopy equivalent to the torus $S^1 \times S^1$.

Then

$$\pi_1(A_n) \cong \mathbb{Z}\langle t \rangle \times F(x_0, \dots, x_n),$$

where t is the class of the framing longitude $S^1 \times \{q\}$ and x_i is a meridian of the i -th strand, the p_i being ordered so that the class of ∂D^2 is $x_0 x_1 \cdots x_n$. The peripheral map $\pi_1(T) \rightarrow P_n$ is

$$(2.1) \quad \lambda \mapsto t, \quad \mu \mapsto x_0 x_1 \cdots x_n,$$

and it is injective.

Deleting the i -th strand fills in the puncture p_i , and doubling it splits p_i into two; hence P_* is the simplicial group $\text{const}(\mathbb{Z}) \times F_*$, where $F_n = F(x_0, \dots, x_n)$ and

$$(2.2) \quad d_i(x_m) = \begin{cases} x_m, & m < i, \\ 1, & m = i, \\ x_{m-1}, & m > i, \end{cases} \quad s_i(x_m) = \begin{cases} x_m, & m < i, \\ x_i x_{i+1}, & m = i, \\ x_{m+1}, & m > i. \end{cases}$$

Lemma 2.1. *The simplicial group F_* of (2.2) is contractible. Consequently, the projection $P_* \rightarrow \text{const}(\mathbb{Z})$ is a weak equivalence of simplicial groups, and*

$$\overline{W}P_* \simeq \overline{W}\text{const}(\mathbb{Z}) = S^1.$$

□

Also, the groups

$$\pi_1(B_n) = \pi_1(B) = \pi_1(S^3 \setminus K)$$

form a constant simplicial group.

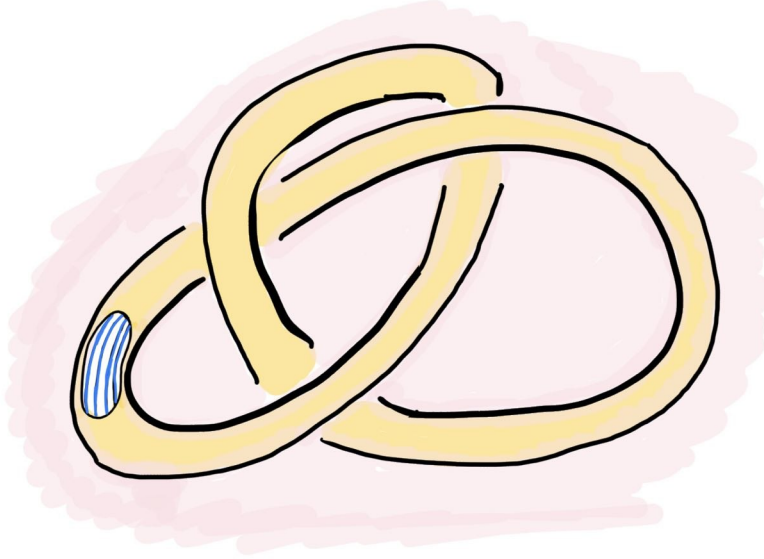
The Seifert–van Kampen theorem applies simplicially: the following pushout computes $\pi_1(S^3 \setminus L_n)$ as a simplicial group.

$$\begin{array}{ccc} \pi_1(A_n \cap B_n) & \hookrightarrow & \pi_1(B_n) \\ \downarrow & & \downarrow \\ \pi_1(A_n) & \longrightarrow & \pi_1(S^3 \setminus L_n) \end{array} \quad = \quad \begin{array}{ccc} \text{const}(\mathbb{Z}^2) & \hookrightarrow & \text{const}(\pi_1(S^3 \setminus K)) \\ \downarrow & & \downarrow \\ \mathbb{Z} \times F_* & \longrightarrow & \pi_1(S^3 \setminus L_n) \end{array}$$

Whitehead’s theorem implies that the homotopy colimit of the corresponding diagram of deloopings is a delooping of the simplicial group $\pi_1(S^3 \setminus L_n)$. By Papakyriakopoulos’ theorem, we may choose $S^3 \setminus K$ as a delooping model for $\text{const}(\pi_1(S^3 \setminus K))$. Thus the pushout diagram takes the following form.

$$\begin{array}{ccc} \overline{W}\text{const}(\mathbb{Z}^2) & \hookrightarrow & \overline{W}(\text{const}(\pi_1(S^3 \setminus K))) \\ \downarrow & & \downarrow \\ \overline{W}(\mathbb{Z} \times F_*) & \longrightarrow & \overline{W}(\pi_1(S^3 \setminus L_n)) \end{array} \quad = \quad \begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\text{in}} & S^3 \setminus K \\ \downarrow \text{pr} & & \downarrow \\ S^1 & \longrightarrow & S^3 \end{array}$$

As a corollary, $\overline{W}\pi_1(S^3 \setminus L_*) = S^3$.



Pink denotes B_n , yellow denotes A_n , and the blue lines are the link components contained in A_n .

References

- [1] Fengchun Lei, Fengling Li, and Jie Wu. “On simplicial resolutions of framed links”. In: *Transactions of the American Mathematical Society* 366.6 (2014), pp. 3075–3093.
- [2] Roman Mikhailov and Inder Bir Singh Passi. *Lower Central and Dimension Series of Groups*. Google-Books-ID: 3mnF84JtM9UC. Springer Science & Business Media, 2009. 367 pp. ISBN: 978-3-540-85817-1.